Lecture 11. Linear Second-Order Equations with Constant Coefficients Part 2

Review: Recall in Lecture 10, we talked about 2nd-order homogeneous equations with constant coefficients of the following form

$$ay'' + by' + cy = 0 \tag{1}$$

To solve for y, we first solve for r from the **characteristic equation**

$$ar^2 + br + c = 0,$$

which has roots $r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Case 1. r_1, r_2 are real and $r_1 \neq r_2$ ($b^2 - 4ac > 0$):

$$\text{General solution: } y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2. r_1, r_2 are real and $r_1 = r_2$ ($b^2 - 4ac = 0$):

$$\text{General solution: } y = (c_1 + c_2 x) e^{r_1 x}$$

In this lecture, we will talk about the last case:

Case 3. r_1, r_2 are complex numbers ($b^2 - 4ac < 0$): (Not covered in Lecture 10)

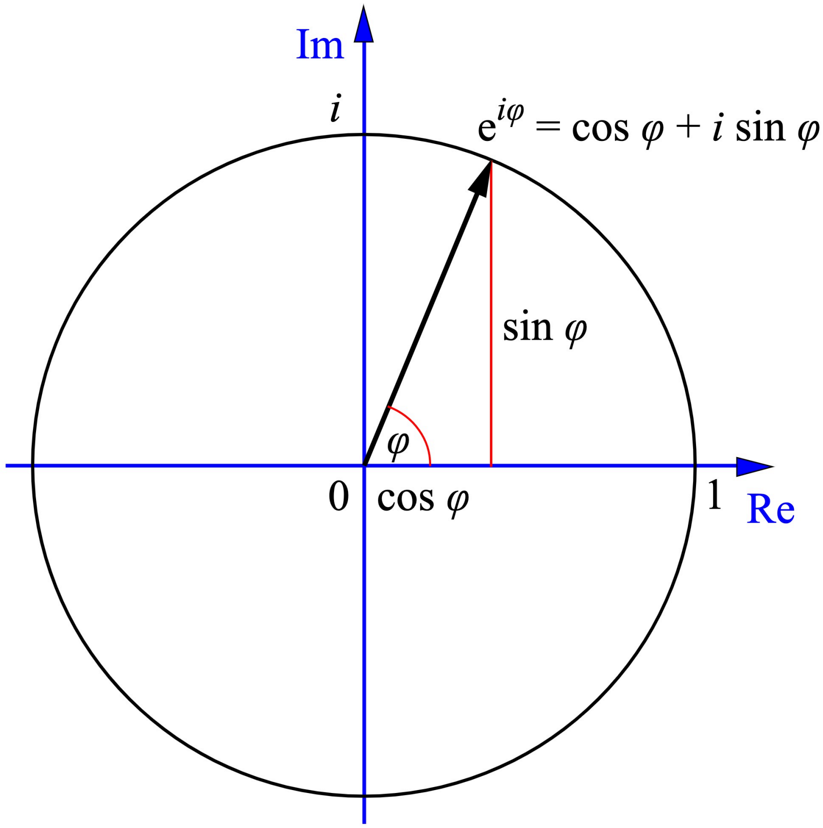
We can write $r_{1,2} = A \pm Bi$.

$$\text{General solution: } y = e^{Ax} \left(c_1 \cos Bx + c_2 \sin Bx \right)$$

Euler's Formula for Complex Numbers

$i = \sqrt{-1}$

- **Euler's formula**: $e^{i\theta} = \cos\theta + i\sin\theta$, $\theta \in \mathbb{R}$



- $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$, where $z = x + iy$ is any complex number.

Theorem 7 Complex Roots

If $r_{1,2} = A \pm Bi$ are roots of the characteristic equation (1), then the corresponding part to the general solution

$$y = e^{Ax}(c_1 \cos Bx + c_2 \sin Bx)$$

Remark: We have the above formula since

$$\begin{aligned}
y(x) &= C_1 e^{r_1 x} + C_2 e^{r_2 x} \\
&= C_1 e^{(A+Bi)x} + C_2 e^{(A-Bi)x} = C_1 e^{Ax} e^{Bix} + C_2 e^{Ax} e^{-Bix} \\
&= C_1 e^{Ax} \cdot (\cos Bx + i\sin Bx) + C_2 e^{Ax}(\cos Bx - i\sin Bx) \\
&= e^{Ax}\left[(C_1 + C_2)\cos Bx + i\,(C_1 - C_2)\sin Bx\right] \\
&= e^{Ax}\left(c_1 \cos Bx + c_2 \sin Bx\right)
\end{aligned}$$

Example 1. Solve the following differential equation:

$$y'' + y' + y = 0$$

ANS: The corresponding char. eqn is

$$r^2 + r + 1 = 0$$

Then

$$r_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3} \cdot \cancel{\sqrt{-1}}\, i}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

By Thm 7, we have the general solution

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right)$$

Example 2. Find the general solution to the homogeneous differential equation

$$\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 125y = 0$$

ANS: The corresponding char. eqn is

$$r^2 - 20r + 125 = 0$$

$$r_{1,2} = \frac{20 \pm \sqrt{20^2 - 4 \times 125}}{2} = \frac{20 \pm \sqrt{-100}}{2} = \frac{20 \pm 10i}{2} = 10 \pm 5i.$$

Thus we have the general solution

$$y(x) = e^{10x}(C_1 \cos 5x + C_2 \sin 5x)$$

Example 3. What values of α and A make $y = A\cos\alpha t$ a solution to $y'' + 7y = 0$ such that $y'(1) = 4$?

ANS: Method 1. Plug the given $y = A\cos\alpha t$ into the eqn with the condition $y'(1) = 4$.

Then solve for α and A.

Method 2. The correspond char. eqn is

$$r^2 + 7 = 0 \Rightarrow r^2 = -7 \Rightarrow r = \pm\sqrt{-7} = \pm\sqrt{7}i = 0 \pm \sqrt{7}i$$

Thus the general solution is

$$y(x) = \cancel{e^{0x}}^{\,1}\left(C_1\cos(\sqrt{7}x) + C_2\sin(\sqrt{7}x)\right)$$

$$\Rightarrow y(x) = C_1\cos\sqrt{7}x + C_2\sin\sqrt{7}x$$

Note if we take $C_2 = 0$, then $y(x) = C_1\cos(\sqrt{7}x)$ is a solution of the one given in the question.

Also, we need to have $y'(1) = 4$.

$$y'(x) = -C_1\sqrt{7}\sin\sqrt{7}x$$

As $y'(1) = 4$, $y'(1) = -C_1\sqrt{7}\sin\sqrt{7} = 4$

$\Rightarrow C_1 = -\dfrac{4}{\sqrt{7}\sin\sqrt{7}}$ Thus $y(x) = -\dfrac{4}{\sqrt{7}\sin\sqrt{7}}\cos\sqrt{7}x$.

is the solution of the form $y = A\cos\alpha x$ satisfies the initial condition. So $A = -\dfrac{4}{\sqrt{7}\sin\sqrt{7}}$ and $\alpha = \sqrt{7}$

Solution using Method 1.

If $y = A\cos\alpha t$, then $y' = -\alpha A\sin\alpha t$ and $y'' = -\alpha^2 A\cos\alpha t$.

Thus, if $y'' + 7y = 0$, then $-\alpha^2 A\cos\alpha t + 7A\cos\alpha t = 0$, so $A\left(7 - \alpha^2\right)\cos\alpha t = 0$.

This is true for all t if $A = 0$, or if $\alpha = \pm\sqrt{7}$. We also have the initial condition: $y'(1) = -\alpha A\sin\alpha = 4$.

Notice that this equation will not work if $A = 0$. If $\alpha = \sqrt{7}$, then $A = -\frac{4}{\sqrt{7}\sin\sqrt{7}}$.

Similarly, if $\alpha = -\sqrt{7}$, we find the same value for A.

Thus, the possible values are $A = -\frac{4}{\sqrt{7}\sin\sqrt{7}}$ and $\alpha = \pm\sqrt{7}$.

Exercise 4. Find y as a function of t if

$$9y'' + 26y = 0,$$

$$y(0) = 2, \quad y'(0) = 4$$

Solution.

The corresponding characteristic equation is

$$9r^2 + 26 = 0.$$

Thus we have

$$r_{1,2} = \pm\frac{i\sqrt{26}}{3}$$

So the general solution is

$$y(x) = c_1\cos\left(\frac{\sqrt{26}x}{3}\right) + c_2\sin\left(\frac{\sqrt{26}x}{3}\right)$$

Substitute $y(0) = 2$ into $y(x) = \cos\left(\frac{\sqrt{26}x}{3}\right)c_1 + \sin\left(\frac{\sqrt{26}x}{3}\right)c_2$, we get $c_1 = 2$

Substitute $y'(0) = 4$ into $y'(x) = -\frac{1}{3}\sqrt{26}\sin\left(\frac{\sqrt{26}x}{3}\right)c_1 + \frac{1}{3}\sqrt{26}\cos\left(\frac{\sqrt{26}x}{3}\right)$:

$$\frac{\sqrt{26}c_2}{3} = 4$$

Thus

$c_1 = 2$

$c_2 = 6\sqrt{\frac{2}{13}}$

Therefore,

$$y(x) = 2\cos\left(\frac{\sqrt{26}x}{3}\right) + 6\sqrt{\frac{2}{13}}\sin\left(\frac{\sqrt{26}x}{3}\right)$$

Exercise 5. (Note this is the case 2 we covered in Lecture 9)

Solve the initial-value problem $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0, y(1) = 0, y'(1) = 1$

Solution.

The corresponding characteristic equation is

$$r^2 + 6y + 9 = 0$$

Thus

$$r_1 = r_2 = -3$$

So we have the general solution

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$$

Substitute $y(1) = 0$ into $y(x)$:

$$\frac{c_1}{e^3} + \frac{c_2}{e^3} = 0$$

Substitute $y'(1) = 1$ into $y' = -3e^{-3x}c_1 + e^{-3x}c_2 - 3e^{-3x}xc_2 :$

$$-\frac{3c_1}{e^3} - \frac{2c_2}{e^3} = 1$$

Soloving the two equations for c_1 and c_2, we have

$c_1 = -e^3$

$c_2 = e^3$

Therefore,

$$y(x) = e^{-3x+3}(x-1)$$